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Course

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Poincaré Lemma in $\mathbb{R}^3$

If $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable scalar field and $g: U \to \mathbb{R}^3$ is a differentiable vector field, the gradient, curl, and divergence are respectively defined as

$$\operatorname{grad}(f) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

$$\operatorname{curl}(g) = \nabla \times g = \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right), \text{ and}$$

$$\operatorname{div}(g) = \nabla \cdot g = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}.$$

Then grad transforms differentiable scalar fields into differentiable vector fields (notation: $\operatorname{grad}: \Omega^0(U) \to \Omega^1(U)$), curl transforms differentiable vector fields into differentiable vector fields (notation: $\operatorname{curl}: \Omega^1(U) \to \Omega^2(U)$), and div transforms differentiable vector fields into scalar fields (notation: $\operatorname{div}: \Omega^2(U) \to \Omega^3(U)$). The elements in the images of grad, curl, and div are known as e*xact forms*. It is obtained that

$$\begin{aligned}
\operatorname{curl}\big(\operatorname{grad}(f)\big) &= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \\
&= (0, 0, 0)
\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}\left(\operatorname{curl}(g)\right) &= \frac{\partial}{\partial x}\left(\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y}\right)\\ &= \frac{\partial^2 g_3}{\partial x\partial y}-\frac{\partial^2 g_2}{\partial x\partial z}+\frac{\partial^2 g_1}{\partial y\partial z}-\frac{\partial^2 g_3}{\partial y\partial x}+\frac{\partial^2 g_2}{\partial z\partial x}-\frac{\partial^2 g_1}{\partial z\partial y}\\ &= \left(\frac{\partial^2 g_3}{\partial x\partial y}-\frac{\partial^2 g_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 g_1}{\partial y\partial z}-\frac{\partial^2 g_1}{\partial z\partial y}\right)+\left(\frac{\partial^2 g_2}{\partial z\partial x}-\frac{\partial^2 g_2}{\partial x\partial z}\right)\\ &= 0\end{aligned}$$

and it is said that $\operatorname{grad}(f)$ and $\operatorname{curl}(g)$ are *closed forms*. Therefore, for every $U \subset \mathbb{R}^3$, every exact form is a closed form.

On the other hand, Henri Poincaré stated (Warner 155) that if U is an open ball of $\mathbb{R}^3$, then there are linear transformations $h_k: \Omega^k(U) \to \Omega^{k-1}(U)$, with $k \in \{1, 2, 3\}$, such that $h_2 \circ \operatorname{curl} + \operatorname{grad} \circ h_1 = \operatorname{id}_{\Omega^1(U)}$ and $h_3 \circ \operatorname{div} + \operatorname{curl} \circ h_2 = \operatorname{id}_{\Omega^2(U)}$. If $\operatorname{curl}(g) = 0$, then $h_2\left(\operatorname{curl}(g)\right) + \operatorname{grad}\left(h_1(g)\right) = g$, which implies that $g = \operatorname{grad}\left(h_1(g)\right)$ is exact. If $\operatorname{div}(g) = 0$, then $h_3\left(\operatorname{div}(g)\right) + \operatorname{curl}\left(h_2(g)\right) = g$, which implies that $g = \operatorname{curl}\left(h_2(g)\right)$ is exact. Therefore, Poincaré Lemma establishes that every closed form on an open ball $U \subset \mathbb{R}^3$ is exact. In physical terms, Poincaré Lemma states that every vector field on an open ball is conservative if it is irrotational and has a vector potential if it solenoidal.

The statement of Poincaré Lemma was done in a higher generality where gradient, curl, and divergence operators are examples of *exterior derivatives* between spaces of differential forms and U is any *simply connected* subset of $\mathbb{R}^n$ for $n \geq 1$. An example of a closed form that is not exact is found when $U = \mathbb{R}^2 - \{(0,0)\} \subset \mathbb{R}^2$: on this non-simply connected space, the exterior derivatives are defined by $df = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ for any scalar field $f: U \to \mathbb{R}$ and $dg = \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}$ for any vector field $g: U \to \mathbb{R}^2$. Let us consider the vector field $g(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$ (Marsden and Tromba 460). Then g is differentiable on U and $dg = 0$. However, there is no differential scalar field $f: U \to \mathbb{R}$ such that $\nabla f = g$: in fact, a candidate scalar field $f: U \to \mathbb{R}$ should be monotone on circular paths around the origin, and this situation would lead to a point $(x, y) \in U$ having two images once a cycle is completed.

Works Cited

Marsden, Jerrold E., and Anthony Tromba. *Vector Calculus*. 6th ed., W. H. Freeman and Company, 2012.

Warner, Frank W. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, 1983.