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Course

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Generalized Stokes Theorem in $\mathbb{R}^n$

The generalized Stokes theorem is a result in multivariable calculus that states (Spivak 102) that if $D \subset \mathbb{R}^n$ is a homeomorphic image of some k-cube $[0, 1]^k = [0, 1] \times \cdots \times [0, 1] \subset \mathbb{R}^k$ and ω is a differentiable form on D, then

$$\int_D d\omega = \int_{\partial D} \omega \quad (\text{Eq. } 1)$$

The existence of a continuous bijective function $\iota : [0, 1]^k \to D$ such that its inverse $\iota^{-1} : D \to [0, 1]^k$ is continuous is assumed so that the existence of a global parametrization on D with $k \leq n$ variables is guaranteed. The set $\partial D = \iota\left(\partial [0, 1]^k \right)$ is the boundary of D, therefore, the image of each face of $[0, 1]^k$ is parametrized by $k - 1$ variables. Forms generalize differentials of functions in single variable calculus: 0-forms are just differentiable scalar fields on D. There are defined directional differentials $dx_1, \ldots, dx_n$ derived from the parametrization ι and every 1-form α is written as a linear combination

$$\alpha = (\alpha_1, \ldots, \alpha_n) \cdot (dx_1, \ldots, dx_n) = \alpha_1 dx_1 + \cdots + \alpha_n dx_n \quad (\text{Eq. } 2)$$

where each α_i is a 0-form. On the basis $\{dx_1, \ldots, dx_n\}$ it is defined a *wedge product* $dx_i dx_j$ that is associative and anticommutative (the latter implies $dx_i dx_i = 0$ for every i). The element $dx_i dx_j$ defines a differential for area, the element $dx_i dx_j dx_l$ defines a differential for volume and so on. Therefore, 2-forms are linear combinations of differentials of area, 3-forms are linear combinations of differentials of volume and so on in the fashion of equation (Eq. 2) (Warner 63). For $k = n$, there is only one non-zero basic differential $dx_1 \cdots dx_n$ of n-volume. The exterior derivative is a linear operator d that transforms $(k-1)$-forms into k-forms and acts on the 0-forms by making

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \qquad (\text{Eq. } 3)$$

For example, if

$\omega = \omega_1 dx_2 dx_3 dx_4 + \omega_2 dx_3 dx_4 dx_1 + \omega_3 dx_4 dx_1 dx_2 + \omega_4 dx_1 dx_2 dx_3$ is a 3-form on $D \subset \mathbb{R}^4$, then $d\omega$ is the 4-form

$$\begin{aligned}
d\omega &= \left(\frac{\partial \omega_1}{\partial x_1} dx_1 + \frac{\partial \omega_1}{\partial x_2} dx_2 + \frac{\partial \omega_1}{\partial x_3} dx_3 + \frac{\partial \omega_1}{\partial x_4} dx_4\right) dx_2 dx_3 dx_4 \\
&\quad + \left(\frac{\partial \omega_2}{\partial x_1} dx_1 + \frac{\partial \omega_2}{\partial x_2} dx_2 + \frac{\partial \omega_2}{\partial x_3} dx_3 + \frac{\partial \omega_2}{\partial x_4} dx_4\right) dx_3 dx_4 dx_1 \\
&\quad + \left(\frac{\partial \omega_3}{\partial x_1} dx_1 + \frac{\partial \omega_3}{\partial x_2} dx_2 + \frac{\partial \omega_3}{\partial x_3} dx_3 + \frac{\partial \omega_3}{\partial x_4} dx_4\right) dx_4 dx_1 dx_2 \\
&\quad + \left(\frac{\partial \omega_4}{\partial x_1} dx_1 + \frac{\partial \omega_4}{\partial x_2} dx_2 + \frac{\partial \omega_4}{\partial x_3} dx_3 + \frac{\partial \omega_4}{\partial x_4} dx_4\right) dx_1 dx_2 dx_3 \\
&= \frac{\partial \omega_1}{\partial x_1} dx_1 dx_2 dx_3 dx_4 + \frac{\partial \omega_2}{\partial x_2} dx_2 dx_3 dx_4 dx_1 + \frac{\partial \omega_3}{\partial x_3} dx_3 dx_4 dx_1 dx_2 + \frac{\partial \omega_4}{\partial x_4} dx_4 dx_1 dx_2 dx_3 \\
&= \left(\frac{\partial \omega_1}{\partial x_1} - \frac{\partial \omega_2}{\partial x_2} + \frac{\partial \omega_3}{\partial x_3} - \frac{\partial \omega_4}{\partial x_4}\right) dx_1 dx_2 dx_3 dx_4
\end{aligned}$$

The generalized Stokes theorem (Eq. 1) summarizes several fundamental results in calculus. If $D = [a, b] \subset \mathbb{R}$ and f is a 0-form on D, then $df = f'(x)dx$ and equation (Eq. 1) becomes

$$\int_{[a,b]} df = \int_{\{a,b\}} f \quad (\text{Eq. } 4)$$

or

$$\int_a^b f'(x)dx = f(b) - f(a) \quad (\text{Eq. } 5)$$

which is the second fundamental theorem of calculus. If $D \subset \mathbb{R}^2$ is homeomorphic to $[0, 1]^2$ and $f = f_1 dx + f_2 dy$ is a 1-form, then $df = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy$ and equation (Eq. 1) becomes

$$\iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \oint_{\partial D} f_1 dx + f_2 dy \quad (\text{Eq. } 6)$$

which is Green's theorem (Marsden and Tromba 431). If $D \subset \mathbb{R}^3$ is a bounded surface homeomorphic to $[0, 1]^2$ and $\omega = f \cdot (dx, dy, dz)$ is a 1-form, then $d\omega = \text{curl}(f) \cdot (dydz, dzdx, dxdy)$ and equation (Eq. 1) becomes

$$\iint_D \operatorname{curl}(f) \cdot (dydz, dzdx, dxdy) = \oint_{\partial D} f \cdot (dx, dy, dz) \qquad (\text{Eq}.\ 7)$$

which is classical Stokes theorem. Furthermore, if $U \subset \mathbb{R}^3$ is homeomorphic to $[0, 1]^3$ and $\omega = f \cdot (dydz, dzdx, dxdy)$ is a 2-form, then $d\omega = \operatorname{div}(f)dxdydz$ and equation (Eq. 1) becomes

$$\iiint_U \operatorname{div}(f)dxdydz = \oiint_{\partial U} f \cdot (dydz, dzdx, dx, dy) \qquad (\text{Eq}.\ 8)$$

which is the divergence theorem.

Works Cited

Marsden, Jerrold E., and Anthony Tromba. *Vector Calculus*. 6th ed., W. H. Freeman and Company, 2012.

Spivak, Michael. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. Addison-Wesley, 1965.

Warner, Frank W. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, 1983.